Name of the Course: Linear Algebra

Syllabus:

Unit I

Symmetric and Skew symmetric matrices - Hermitian and Skew Hermitian Matrices – Orthogonal and Unitary matrices - Rank of matrix - Eigen values and Eigen vectors of Linear operators – Cayley Hamilton theorem - Solutions of Homogeneous linear equations – Solutions of non homogenuous linear equations.

Section 1.5 Eigen values and Eigen vectors of the Linear Operators . (Or) Characteristic values and Characteristic vectors of a given matrix.

Definition: 1.5.1 :

Let $A = (a_{ij})$ be an n rowed square matrix and λ be a scalar. Then the matrix $A - \lambda I$ is called **characteristic matrix of A**. Also the determinant $[A - \lambda I] = 0$ is called an ordinary polynomial of λ of degree n and is **characteristic polynomial** The solution of the equation $[A - \lambda I] = 0$ are called **Characteristic values or eigen values** of the matrix A.

Eigen vectors or Characteristic vectors.

The solution X of the equation $(A - \lambda I)X = 0$ corresponding the eigen values λ are called **Characteristic vectors or eigen vectors** of the matrix A.

Problem: 1.5.2

Simplify Find the characteristic polynomial and the eigen values of the matrix $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix}$.

Solution:

Characteristic equations is $|(A - \lambda I)| = 0$

$$\left| \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix} = 0$$

$-\lambda\,(\lambda^2-1)-1(-\lambda+2\,)+2\,(-1+2\lambda)=\;0$

$-\lambda^3+\lambda+\lambda-2-2\;+4\lambda=\;0$

$-\lambda^3+\;6\lambda-4=0.$

Expanding this determinant, we get $\lambda^3-\;6\lambda+4=0.$

$(\lambda-2)(\,\lambda^2+\;2\lambda-2)=\;0$

Solving we get, $\lambda=2$ and $\lambda=-1\pm\sqrt{3}$. The eigen values of the matrix A are

$2,-1+\sqrt{3}\,,-1-\sqrt{3}.$

Problem: 1.5.3

Find the characteristic polynomial and the eigen values of the matrix $A=\begin{pmatrix}1&-1&2\\-2&1&3\\3&2&-3\end{pmatrix}$

Solution: $\lambda=2$, $\lambda=\frac{-3}{2}+\frac{\sqrt{61}}{2}$, $\lambda=\frac{-3}{2}-\frac{\sqrt{61}}{2}$

Short Cut Method:

$\lambda^3-\;\lambda^2(Sum\ of\ the\ diagonal\ selements)+$
$\lambda\,(Sum\ of\ the\ co\ factors\ \ of\ diagonal\ elements)-\;\det A=0$

Problem: 1.5.4

Determine the eigen values and eigen vectors of the matrix $A=\begin{pmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{pmatrix}$

Solution:

Sum of the diagonal elements of A is 8+7+3 = 18

$Sum\ of\ the\ co\ factors\ \ of\ diagonal\ elements$ is

$$\begin{vmatrix}7&-4\\-4&3\end{vmatrix}+\begin{vmatrix}8&2\\2&3\end{vmatrix}+\begin{vmatrix}8&-6\\-6&7\end{vmatrix}\;=\;5+20+20=45$$

$$detA=\begin{vmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{vmatrix}=8(21-16)+6\,(-18+8)+2(24-14)=\;40-60+20=0$$

$\lambda^3-\;\lambda^2(18)+\lambda\,(5+20+20)-\;0=0$

$\lambda^3-\;18\lambda^2+45\lambda\;=0$

$\lambda(\lambda^2-\;18\lambda+45)\;=0$

$$\lambda(\lambda-3)\,(\lambda-15)=0$$

$$\lambda=0,\ \lambda=3,\ \lambda=\ 15$$

The eigen values of the matrix A are $0, 3, 15$.

To find Eigen vectors:

$(A-\ \lambda\, I)X=0$ where $X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ is

$$\left(\begin{pmatrix}8 & -6 & 2\\-6 & 7 & -4\\2 & -4 & 3\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\right)\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$$

$$\begin{pmatrix}8-\lambda & -6 & 2\\-6 & 7-\lambda & -4\\2 & -4 & 3-\lambda\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \quad \text{.................. (1)}$$

Case 1 : Put $\lambda=0$ in equation (1) , we get

$$\begin{pmatrix}8 & -6 & 2\\-6 & 7 & -4\\2 & -4 & 3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$$

$8x_1-\ 6x_2+\ 2x_3=0$ .... (2)

$-6x_1+7x_2-4x_3=0$ .... (3)

$2x_1-\ 4x_2+\ 3x_3=0$ .... (4)

(3) $-6x_1+7x_2-4x_3=0$

3 (4) $\Longrightarrow$ $6x_1-\ 12x_2+\ 9x_3=0$

Adding the above two results , we get $-5x_2+5x_3=0 \ \Longrightarrow\ x_2=x_3$

$x_3=\ x_2$ in equation (2), we get

$8x_1-\ 6x_2+\ 2x_2=0 \quad \Longrightarrow 8x_1=\ 4x_2$.

Put $x_3=\ x_2=1$ in the above result, we get, $8x_1=\ 4(1) \ \Longrightarrow\ x_1=1/2$

Therefore, Eigen vector is $X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1/2\\1\\1\end{pmatrix}$ *or* $\begin{pmatrix}1\\2\\2\end{pmatrix}$

Case 2 : Put $\lambda=3$ in equation (1) we get,

$$\begin{pmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$5x_1 - 6x_2 + 2x_3 = 0$ .... (2)

$-6x_1 + 4x_2 - 4x_3 = 0$ .... (3)

$2x_1 - 4x_2 + 0x_3 = 0$ .... (4)

From the equation (4) we get , $2x_1 = 4x_2 \implies x_1 = 2x_2$

(3) $-6x_1 + 4x_2 - 4x_3 = 0$

3 (4) $\implies$ $6x_1 - 12x_2 + 0x_3 = 0$

Adding the above two results , we get $-8x_2 - 4x_3 = 0 \implies -8x_2 = 4x_3$

$-8x_2 = 4x_3 \implies x_3 = -2x_2 \implies 2x_2 = -x_3$

From the results $x_1 = 2x_2$ and $2x_2 = -x_3$, we get $x_1 = -x_3$

$x_1 = 2x_2 \implies \frac{x_1}{2} = \frac{x_2}{1}$ and $2x_2 = -x_3 \implies \frac{x_1}{1} = \frac{x_3}{-2}$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

Therefore, Eigen vector is $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

Case 3: Put $\lambda = 15$ in equation (1) we get,

$$\begin{pmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{.................. (1)}$$

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & 3-12 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$-7x_1 - 6x_2 + 2x_3 = 0$ .... (2)

$-6x_1 - 8x_2 - 4x_3 = 0$ .... (3)

$2x_1 - 4x_2 - 12x_3 = 0$ .... (4)

(3) $\quad -6x_1 - 8x_2 - 4x_3 = 0$

3 (4) $\Longrightarrow \quad 6x_1 - 12x_2 - 36x_3 = 0$

Adding the above two results , we get $-20x_2 - 40x_3 = 0 \Longrightarrow 20x_2 = -40x_3 \Longrightarrow x_2 = -2x_3$

$Put\ x_2 = -2x_3$ in equation (2), we get

$-7x_1 - 6(-2x_3) + 2x_3 = 0$ we get $-7x_1 = 14x_3 \Longrightarrow x_1 = -2x_3$

$x_2 = -2x_3$ and $x_1 = -2x_3$ gives $x_1 = x_2$. Put $x_1 = x_2 = 1$ we get $x_3 = -1/2$

Therefore, Eigen vector is $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1/2 \end{pmatrix}$ or $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$